BOHR COMPACTIFICATIONS AND A RESULT OF FØLNER

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ABSTRACT

In this paper, we study the Bohr compactification of an arbitrary topological group T with regard to obtaining relations between relatively dense (or discretely syndetic) subsets of T, and neighborhoods of the identity in the Bohr compactification. The methods utilized are those algebraic techniques which have been recently applied to topological dynamics (see [2]). For an abelian group, we show that cls $(A^{-1}AAa^{-1})$, for A relatively dense and $a \in A$, is usually a neighborhood of the identity, thus generalizing a result of Følner [4]. Moreover, an analogous result is proved in the non-abelian case under additional assumptions. Finally, we utilize these results to obtain a generalization of a result of Cotlar-Ricabarra [1] concerning maximal almost periodicity in abelian topological groups.

1. Introduction

The major goal of this paper is to make a systematic and unified study of the Bohr compactification of an arbitrary topological group T. The methods to be used are the algebraic techniques developed by Ellis in the study of topological dynamics (see [2] for a general account); in particular, the techniques used in [3] to study equicontinuity will be utilized extensively. The impetus for this study came from an attempt to recover and generalize Følner's topological assertions in [4] concerning relatively dense (or discretely syndetic) subsets of the topological group and neighborhoods of the identity in the Bohr compactification.

The methods involved in Følner's paper are basically analytic, involving substantial use of Banach means on the original group and Fourier analysis. In this paper, we avoid the use of Fourier analysis by looking at an algebraic represen-

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tation of the Bohr compactification which is particularly amenable to analysis via finite-dimensional representations. The thrust of this study is to provide generalizations in two directions. First, we obtain some more precise results in the abelian case. Second, we do obtain a non-abelian theory which is based on the asumption that the Bohr compactification does assume the desired form (this is always obtained in the abelian case). We will comment further on this assumption.

To be more precise, the following result is obtained in [4].

THEOREM. *Let T be an abelian topological group, A a discretely syndetic subset of T, and V a neighborhood of the identity e. Then there exist characters $\chi_1,\cdots,\chi_n$ such that if $t\in T$ and* $\operatorname{Re}\chi_r(t) \geqq 0$ $(r = 1,\cdots,n)$, *then $t\in AA^{-1}AA^{-1}V$.*

The proof involves analyzing, via the translation invariant mean dominated by the Banach upper mean, the characteristic function of a uniform open swelling AV_0 of A, and obtaining a positive definite function. The main result invoked is Godement's decomposition, on locally compact groups, of such a function into an almost periodic function and a function h with $|h|^2$ having 0 mean value. The proof proceeds by estimating the Fourier coefficients of the almost periodic function. In our paper, we obtain the following generalization: *If A is a discretely syndetic subset of an abelian topological group T, then there exist continuous characters $\chi_1,\cdots,\chi_n$ and $\varepsilon > 0$ such that $t\in T$ with $|\chi_r(t) - 1| < \varepsilon$ $(1 \leqq r \leqq n)$ implies $t\in \operatorname{cls}(A^{-1}AAa^{-1})$ for "most" $a\in A$.* In addition to reducing the number of times A is used from 4 to 3, it indicates that the characters can be picked independent of neighborhoods of e.

The methods employed also give results for non-abelian T, viz: *If A is discretely syndetic in a topological group T, then there exists continuous finite-dimensional unitary representations $\chi_1,\cdots,\chi_m$ and $\varepsilon > 0$ such that $t\in T$ with $\|\chi_r(t) - I\| < \varepsilon$ $(1 \leqq r \leqq m)$ implies $t\in A^{-1}AAVA^{-1}$ for every neighborhood V of e.* However, in this case the basic assumption of our approach ($H(G,\mathscr{R}) \supset E$; see below) is not automatically true as it is for abelian groups T. Indeed it is not an easy assumption to verify in general, and in particular we do not know whether it is true for amenable groups, T. Thus the Fourier analysis techniques give results for amenable non-abelian groups which we are unable to obtain. The non-abelian theory is not devoid of interest, however, for the following reason. Let (X,T) be a compact minimal transformation group, (Y,T) its maximal equicontinuous factor, and $\phi:(X,T) \xrightarrow{\sim} (Y,T)$ the canonical map. Let $x_0\in X$ and $y_0 = \phi(x_0)$. Then the maps $t\to x_0t$: $T\to X$ and $t \to y_0t$: $T\to Y$ induce topologies $\mathscr{T}$ and $\mathscr{S}$ respectively on T. The various results about the Bohr compactification of T discussed here may be viewed as results concerning the relationship between $\mathscr{T}$ and $\mathscr{S}$ (When X is taken to be the universal minimal set for T, then Y is just the Bohr compactification). Now the methods used in this paper may be applied to the above more general situation to give similar results. Here simply verifiable conditions (e.g., point distal) may be imposed on (X,T) to guarantee that the requisite assumptions be satisfied.

Finally we consider the Cotlar-Ricabarra [1] result for maximal almost periodicity, which states that in an abelian topological group T, if $s \in T$ with $s \neq e$, then there exists a continuous character χ with $\chi(s) \neq 1$ iff $s \notin U^6$ for some discrete syndetic symmetric open neighborhood U of e. We obtain U^4 easily, and, in fact, $\mathrm{cls}(U^3 x)$. We believe that the methods of this paper can be pushed to yield U^3, the best obtainable result.

We shall assume the notation of [2] in general. However, for the sake of completeness, we include the following brief summary of the relevant facts. Thus $(T,\mathscr{T})$ will denote an arbitrary Hausdorff topological group, and βT the Beta-compactification of the discrete underlying group T. Also, $\mathscr{C}$ will denote the continuous functions on βT. $\mathscr{C}$ is, of course, canonically isomorphic to the bounded functions on T. Recall that there is a correspondence between pointed transformation groups (i.e., having a distinguished point with dense orbit) and T-subalgebras $\mathscr{A} \subset \mathscr{C}$, i.e., $\mathscr{A}$ is a uniformly closed subalgebra of $\mathscr{C}$ such that $f \in \mathscr{A}$ implies $tf \in \mathscr{A}$ $(t \in T)$, where $\langle tf, s\rangle = \langle f, st\rangle$. If (X, x_0, T) is pointed and $\mathscr{A}$ is an algebra corresponding to it (in general, there are many such algebras), then $\mathscr{A}$ is isomorphic to $\mathscr{C}(X)$. There are many universal minimal sets for T discrete in βT; we shall distinguish one and denote it by M. These minimal sets correspond to minimal right ideals of the semigroup βT. From the general theory of these semigroups, it follows that there exist idempotents in M, and we again distinguish one and denote it by u. We then have that $G = Mu$ is a group. We let J denote the set of idempotents in M.

It turns out that an algebra corresponding to M is $\mathfrak{A}(u) = \{f \mid f \in \mathscr{C}, fu = f\}$, and that all minimal sets of T have representative algebras contained in $\mathfrak{A}(u)$. Thus, we let $\mathscr{E}$ be the unique representative algebra for the universal equicontinuous minimal set (there is only one in this case). $\mathscr{E}$ is the algebra of almost periodic functions [2, Proposition 15.7].

In general, the way that one brings the topology $\mathscr{T}$ on T into the algebraic picture is to consider the T-subalgebra $\mathscr{R}$ of bounded right uniformly continuous functions on $(T, \mathscr{T})$ [2; 2, 3 of Notes of Chapter 9]. It turns out that the minimal version of $\mathscr{R}$, namely $\mathscr{R}_u = \mathscr{R} \cap \mathfrak{A}(u)$, is sufficient for our purposes.

With every T-subalgebra $\mathscr{A}$ of $\mathfrak{A}(u)$, we associate a group $\mathfrak{g}(\mathscr{A}) = A = \{\alpha \in G \mid f\alpha = f (f \in \mathscr{A})\}$. Thus, we will consider the groups $E = \mathfrak{g}(\mathscr{E})$ and $R = \mathfrak{g}(\mathscr{R}_u)$.

If $\mathscr{A}$ is a T-subalgebra of $\mathfrak{A}(u)$, then $\mathscr{A}$ induces a topology $\tau(\mathscr{A})$ on G as follows: if $K \subset G$, then $\alpha \in \mathrm{cls}_{\tau(\mathscr{A})}K$ iff $f^{\alpha} \leqq f^{K}u$ ($f \in \mathscr{A}_R$, the real-valued functions in $\mathscr{A}$). We have that $(G, \tau(\mathscr{A}))$ is always compact but, in general, satisfies none of the separation axioms. We will consider $\tau(\mathscr{E})$ and $\tau(\mathscr{R}_u)$ in this paper.

Finally, we shall frequently use the notations of [3].

2. Bohr Compactifications

NOTATION (2.1). We will denote by $B(T,\mathscr{T})$ the transformation group $|\mathscr{E} \cap \mathscr{R}|$. Note that $\mathscr{E} \cap \mathscr{R} = \mathscr{E} \cap \mathscr{R}_u$, since $\mathscr{E} = \mathscr{E} \cap \mathfrak{A}(u)$ [2, Prop. 15.7], and thus is a minimal subalgebra.

REMARK (2.2). $B(T,\mathscr{T})$ is a compact topological group for which $(B(T,\mathscr{T}),(T,\mathscr{T}))$ is a jointly continuous transformation group with the action induced by a homorphism (not necessarily injective) with dense image. Moreover, if G is a compact topological group on which $(T,\mathscr{T})$ acts in a jointly continuous fashion via a homomorphism with dense image, then there exists a transformation group homomorphism $\phi\colon (B(T,\mathscr{T}),(T,\mathscr{T})) \tilde{\to} (G, T)$ (thus ϕ is a group homomorphism). Thus, $B(T,\mathscr{T})$ is the Bohr compactification.

DEFINITION (2.3). Let χ be a bounded continuous finite-dimensional representation of $(T,\mathscr{T})$. If $G_\chi = \mathrm{cls}(\chi(T))$, then G_χ is a compact topological group satisfying the condition of Remark (2.2) Let $\mathscr{A}_\chi \subset \mathfrak{A}(u)$ be an algebra such that $(|\mathscr{A}_\chi|, (T,\mathscr{T})) \simeq (G_\chi,(T,\mathscr{T}))$. Then $\mathscr{A}_\chi$ is called a *representative algebra for* χ.

To see that G_χ is a compact topological group, it is sufficient to show that G_χ is closed in the appropriate version of $\mathbb{C}^{n^2}$, as $\|\chi(t)\| \leqq M$ for some fixed M. Now it is clear that $\det(\chi(t))$ is also uniformly bounded (in some polynomial of M). If $|\det(\chi(t)| \neq 1$, then either $|\det(\chi(t))|$ or $|\det(\chi(t^{-1}))| > 1$, say the former. Then $|\det(\chi(t^n))| \to \infty$, a contradiction. Hence, $|\det(\chi(t))| = 1$, and $\chi(T) \subset SL(n, \mathbb{C})$, which is closed in $\mathbb{C}^{n^2}$. This means that $G_\chi = \mathrm{cls}\,\chi(T)$ is also closed in $\mathbb{C}^{n^2}$.

All representations considered will be bounded.

Recall that if $(\mathscr{B}_\alpha)$ is a family of T-subalgebras of $\mathscr{C}$, then $\bigvee \mathscr{B}_\alpha$ is the least T-subalgebra containing all the $\mathscr{B}_\alpha$'s.

LEMMA (2.4). *$\mathscr{E} \cap \mathscr{R} = \bigvee\{\mathscr{A}_\chi \mid \chi$ a continuous finite dimensional representation of $(T,\mathscr{T})\}$.*

PROOF. First notice that $\mathscr{R}p \subset \mathscr{R}$ $(p \in \beta T)$ and the fact that $\mathscr{E}$ is regular [2, Definition 11.19.1] implies $\mathscr{E} \cap \mathscr{R}$ is regular. It then follows that for every χ, $\mathscr{A}_\chi \subset \mathscr{E} \cap \mathscr{R}$ and thus $\bigvee \mathscr{A}_\chi \subset \mathscr{E} \cap \mathscr{R}$.

Next consider the topology $\mathscr{T}_1$ induced on T by considering the orbit of the identity $e \in |\mathscr{E} \cap \mathscr{R}|$. Since the canonical map $\pi:(T,\mathscr{T}) \to |\mathscr{E} \cap \mathscr{R}|$ is continuous, we have that $\mathscr{T} \supset \mathscr{T}_1$. Moreover, since $(\pi(T),(\mathscr{T}_1))$ is a dense subgroup of the compact group $|\mathscr{E} \cap \mathscr{R}|$, it follows that the continuous finite-dimensional representations of $(\pi(T),(\mathscr{T}_1))$ separate points of $|\mathscr{E} \cap \mathscr{R}|$. It then follows that the continuous finite-dimensional representations of $(T,\mathscr{T}_1)$ separate points of $|\mathscr{E} \cap \mathscr{R}|$ and, since $\mathscr{T} \supset \mathscr{T}_1$, the continuous finite-dimensional representations of $(T,\mathscr{T})$ separate points. Again by the regularity of $\mathscr{E} \cap \mathscr{R}$, $\bigvee \mathscr{A}_\chi \supset \mathscr{E} \cap \mathscr{R}$, and thus $\bigvee \mathscr{A}_\chi = \mathscr{E} \cap \mathscr{R}$.

LEMMA (2.5). *$\tau(\mathscr{R}) = \tau(\mathscr{R}_u)$.*

PROOF. We first show that $\mathscr{R}u = \mathscr{R} \cap \mathfrak{A}(u)$. Clearly $\mathscr{R} \cap \mathfrak{A}(u) \subset \mathscr{R}u$, and since $\mathscr{R}u \subset \mathscr{R}$, then $\mathscr{R}u \subset \mathscr{R} \cap \mathfrak{A}(u)$.

Finally, $\tau(\mathscr{R}) = \tau(\mathscr{R}u) = \tau(\mathscr{R} \cap \mathfrak{A}(u)) = \tau(\mathscr{R}_u)$ by [2, 4 of (11.11)].

The next result provides the first identification of $|\mathscr{E} \cap \mathscr{R}|$. This holds independent of any commutativity assumptions on T

LEMMA (2.6). *$|\mathscr{E} \cap \mathscr{R}| \simeq (G/ER, \tau(\mathscr{E}))$.*

PROOF. Consider the canonical map $|\mathscr{E}| \xrightarrow{\sim} |\mathscr{E} \cap \mathscr{R}_u|$. Now $|\mathscr{E}| \simeq (G/E, \tau(\mathscr{E}))$ by [2, (15.8)]. This induces a homomorphism $\phi:(G/E,\tau(\mathscr{E})) \to |\mathscr{E} \cap \mathscr{R}_u|$ given by $\phi(Epu) = p|\mathscr{E} \cap \mathscr{R}_u$.

Suppose $\phi(Epu) = u \mid \mathscr{E} \cap \mathscr{R}_u$. Then $p \mid \mathscr{E} \cap \mathscr{R}_u = u \mid \mathscr{E} \cap \mathscr{R}_u$, and $pu \in \mathfrak{g}(\mathscr{E} \cap \mathscr{R}_u) = ER$ [2, (14.16)]. Next, let $\gamma \in E$, $\delta \in R$. Then ϕ $(E\gamma\delta u)$ $=\gamma\delta | \mathscr{E} \cap \mathscr{R}_u = u | \mathscr{E} \cap \mathscr{R}_u$. It follows from these remarks that $|\mathscr{E} \cap \mathscr{R}_u| = ((G/E)/(ER/E),\ \tau(\mathscr{E})) = (G/ER, \tau(\mathscr{E}))$, as desired.

Since the results of [2, (15.6) to (15.8)] also show that $|\mathscr{E}| \simeq (G/E, \tau(\mathscr{D}))$, it also follows that $|\mathscr{E} \cap \mathscr{R}_u| \simeq (G/ER, \tau(\mathscr{D}))$.

The major problem with the above characterization is that the role of $\mathcal{T}$ via the algebra $\mathcal{R}$ is obscured. The next result settles this problem.

LEMMA (2.7).

1. *The canonical map* $\pi:(G,\tau(\mathcal{E}\cap\mathcal{R}))\to(G/ER,\tau(\mathcal{E}))$ *is continuous.*
2. $|\mathcal{E}\cap\mathcal{R}|\simeq(G/ER,\tau(\mathcal{E}\cap\mathcal{R}))\simeq(G/ER,\tau(\mathcal{R}))$.

PROOF. 1. It is equivalent by (2.6) to show that $\pi_1:(G,\tau(\mathcal{E}\cap\mathcal{R}))\to|\mathcal{E}\cap\mathcal{R}|$, $\gamma\to\gamma|\mathcal{E}\cap\mathcal{R}$, is continuous. Since $\mathcal{C}(|\mathcal{E}\cap\mathcal{R}|)=\mathcal{E}\cap\mathcal{R}$, it is sufficient to show that $f\in\mathcal{E}\cap\mathcal{R}$ implies $f\pi_1$ is continuous on $(G,\tau(\mathcal{E}\cap\mathcal{R}))$. But by [2, Lemma 14.12] with $\mathcal{F}=\mathbb{C}\subset\mathcal{E}\cap\mathcal{R}\subset\mathcal{E}=\mathcal{F}^{\#}$, it is immediate that $f\pi_1$ is continuous on $(G,\tau(\mathcal{E}\cap\mathcal{R}))$. The result follows.

2. Since $(G,\tau(\mathcal{E}\cap\mathcal{R}))$ is compact and $(G/ER,\tau(\mathcal{E}))$ is compact Hausdorff, it follows by 1, that $(G/ER,\tau(\mathcal{E}\cap\mathcal{R}))\simeq(G/ER,\tau(\mathcal{E}))$. Moreover, since $(G/ER,\tau(\mathcal{R}))$ is compact and $\tau(\mathcal{E}\cap\mathcal{R})\subset\tau(\mathcal{R})$ on G/ER, we also have that $(G/ER,\tau(\mathcal{E}\cap\mathcal{R}))\simeq(G/ER,\tau(\mathcal{R}))$

Note that if T is discrete, $\mathcal{R}=\mathcal{C}$ and we have that the Bohr compactification is $(G/E,\tau)$, in agreement with [2, Prop. 15.8]. Also the canonical homomorphism from T into $(G/ER,\tau(\mathcal{R}))$ is $t\to ER(utu)$.

The next result is the desired identification of $B(T,\mathcal{T})$ when T is abelian. The idea is to characterize ER in a form in which the role of $\mathcal{R}$ is transparent and which will enable us to do further analysis.

Recall that $H(G,\mathcal{R})=\cap\{\mathrm{cls}_{\tau(\mathcal{R})}V\mid V \text{ a } \tau(\mathcal{R})\text{-neighborhood of } u\}$, [2, Remark 14.10]. Note that, of course, $\mathcal{R}\nsubseteq\mathfrak{A}(u)$, but all the comments are still applicable, since $\tau(\mathcal{R})=\tau(\mathcal{R}_u)$.

THEOREM (2.8). *If T is abelian, then* $B(T,\mathcal{T})\simeq(G/H(G,\mathcal{R}),\tau(\mathcal{R}))$.

PROOF. By 2.7, it suffices to show that $H(G,\mathcal{R})=ER$. Since $\tau(\mathcal{R})\subset\tau$, it follows by [2, 15.11] that $H(G,\mathcal{R})\supset\cap\{\mathrm{cls}_\tau W\mid W \text{ a } \tau\text{-neighborhood of } u\}=E$ (the latter equality is known at this point only for T abelian). Moreover, clearly $H(G,\mathcal{R})\supset R$. Finally, since $\mathcal{R}\alpha\subset\mathcal{R}(\alpha\in G)$, then $H(G,\mathcal{R})$ is a subgroup of G and thus $H(G,\mathcal{R})\supset ER$.

Next, consider the canonical map $\pi:(G,\tau(\mathcal{R}))\to(G/ER,\tau(\mathcal{R}))$. Let U be a closed neighborhood of $\pi(u)$. Then $\pi(W)\subset U$ for some $\tau(\mathcal{R})$-neighborhood W of u, and $\pi(\mathrm{cls}_{\tau(\mathcal{R})}W)\subset\mathrm{cls}_{\tau(\mathcal{R})}\pi(W)\subset\mathrm{cls}_{\tau(\mathcal{R})}U=U$. Thus, $\pi(H)\subset\pi(\mathrm{cl}_{\tau(\mathcal{R})}W)\subset U$, whence $\pi(H)\subset\cap\{U\mid U \text{ a } \tau(\mathcal{R})\text{-neighborhood of } \pi(u)\}=\pi(u)$, since $(G/ER,\tau(\mathcal{R})$ is Hausdorff. This means that $H(G,\mathcal{R})\subset ER$.

With regard to the assumption $H(G, \mathscr{R}) = ER$, we have already noted that the assumption $H(G,\tau)$ $(= \cap \{\mathrm{cls}_\tau W \mid W a \tau\text{-neighborhood of } u\}) = E$ is sufficient. This latter assumption is false in general. The group $T = \mathrm{SL}(2, \mathbb{R})$, provides a counter-example. This is partially due to the facts that there exist non-trivial proximal flows for $SL\ (2, \mathbb{R})$. i.e., minimal flows for which every pair of points get arbitrarily close under the group action, and that there are no almost periodic functions on $\mathrm{SL}(2, \mathbb{R})$. It is conjectured that if the group T does not admit any non-trivial minimal proximal flows, then $H(G,\tau) = E$. (N.B. This conjecture has recently been verified. Since it is known that all nilpotent groups satisfy the hypothesis, the following theory holds in this case). Of course it is possible that $H(G,\tau) \neq E$ but $H(G,\mathscr{R}) = ER$. However, no conditions are presently known which enable one to directly compute $H(G,\mathscr{R})$ without computing $H(G,\tau)$.

3. Generalizations of Følner's results

In this section, we will develop a unified approach to Følner's results. The standing assumption on $(T,\mathscr{T})$ ($\mathscr{T}$ will always stand for a Hausdorff topological group topology for T) is that $B(T,\mathscr{T}) \simeq (G/H(G,\mathscr{R}),\tau(\mathscr{R}))$. We also let $\mathscr{N}_{(\ )}(\mathscr{T})$ denote a neighborhood system. Also recall that if $A \subset T$, we define $h(A) = \{p \mid p \in \beta T$ and $A \in p\}$. Thus $h(A) \cap G = \{\alpha \mid \alpha \in G$ and $A \in \alpha\}$, which will be denoted $h_G(A)$.

LEMMA (3.1). *Consider $(T,\mathscr{T})$ and let $A \subset T$, $U \in \mathscr{N}_e(\mathscr{T})$, where e is the identity of T. Then there exists $f\colon T \to [0,1]$ uniformly continuous such that $f \mid A \equiv 0$, $f \mid T - AU \equiv 1$.*

PROOF. It is well-known that we can find $g\colon T \to [0,1]$ uniformly continuous such that $g(e) = 1$, $g \mid T - U \equiv 0$. Now define $h(x) = \sup_{y \in A} g(xy^{-1})$. It is direct to verify that $h \mid A \equiv 1$, $h \mid T - AU \equiv 0$ and h is uniformly continuous. Finally, let $f = 1 - h$.

Suppose that we have a continuous finite-dimensional representation χ of $(T,\mathscr{T})$. Then as a map on T with its discrete topology, χ has a continuous extension $\hat{\chi}$ to βT.

LEMMA (3.2). *$\hat{\chi}$ is a homomorphism of βT.*

PROOF. Let $p \in \beta T$, (t_κ) a net in T with $t_\kappa \to p$. If $t \in T$, then $\hat{\chi}(pt) = \hat{\chi}(\lim t_\kappa t) = \lim \hat{\chi}(t_\kappa t) = \lim\ (\hat{\chi}(t_\kappa)\hat{\chi}(t)) = [\lim \hat{\chi}(t_\kappa)]\hat{\chi}(t) = \hat{\chi}(p)\hat{\chi}(t)$.

Now if $q, r \in \beta T$ and (t_ν) a net in T with $t_\nu \to r$, $\hat{\chi}(\lim qt_\nu) = \lim \hat{\chi}(qt_\nu) = \lim \hat{\chi}(q)\hat{\chi}(t_\nu)$ $= \hat{\chi}(q)\hat{\chi}(r)$.

For the next lemma, recall that if $\phi \neq K \subset \beta T$ and $f \in \mathscr{C}_R$, then $f^K \in \mathscr{C}$ is defined by $\langle f^K, t\rangle = \sup_{k \in K} \langle fk, t\rangle$.

LEMMA (3.3). *Let $C \subset T \subset \beta T$ and $f \in \mathscr{C}_R$, the real-valued functions. Then* $f^{h_G(C)} \leqq f^{h(C)} = f^C$ *if* $h_G(C) \neq \varnothing$.

PROOF. Since $h_G(C) \subset h(C)$, the first inequality follows by [2, 11.8]. Now let $p \in h(C)$ and $t \in T$. Since $h(C)$ is a βT-open neighborhood of p, there exists a net $(s_\kappa) \in C$ (equivalently, $h(s_\kappa) \in h(C)$) such that $s_\kappa \to p$. Then $s_\kappa t \to pt$ and $\langle fp, t\rangle = \langle f, pt\rangle = \lim \langle f, s_\kappa t\rangle = \lim \langle fs_\kappa, t\rangle \leqq \sup_C \langle fs, t\rangle = \langle f^C, t\rangle$. Consequently $\langle f^{h(C)}, t\rangle = \sup_{h(C)} \langle fq, t\rangle \leqq \langle f^C, t\rangle$, and so $f^{h(C)} \leqq f^C$. Since $C \subseteq h(C)$. we have $f^{h(C)} = f^C$.

The following result is crucial to our analysis.

LEMMA (3.4). *Let $B \subset T$, $C \in u$. Then* $\mathrm{cls}_{\tau(\mathscr{R})} h_G(B) \subset h_G(BCV)$ *for every* $V \in \mathscr{N}_e(\mathscr{T})$.

PROOF. Let $V \in \mathscr{N}_e(\mathscr{T})$. If $BCV = T$, the result is obvious. So assume $BCV \neq T$. By (3.1), there exists $f: (T, \mathscr{T}) \to [0,1]$ uniformly continuous such that $f \mid BC \equiv 0$, $f \mid T - BCV \equiv 1$.

Now let $\alpha \in \mathrm{cls}_{\tau(\mathscr{R})} h_G(B)$. By the definition of the $\tau(\mathscr{R})$-topology $f\alpha \leqq f^{h_G(B)}u$. Then by (3.3), $f\alpha \leqq f^B u$, and $\langle f, \alpha\rangle = \langle f\alpha, e\rangle \leqq \langle f^B u, e\rangle = \langle f^B, u\rangle$. If $s \in B$, $r \in C$, then $\langle fs, r\rangle = \langle f, sr\rangle = 0$, since $sr \in BC$. Now $C \in u$ implies that there exists a net $(t_\kappa) \in C$ such that $t_\kappa \to u$ in the ordinary topology of βT. It follows that $\langle f^B, t_\kappa\rangle \to \langle f^B, u\rangle$. Since $\langle f^B, t_\kappa\rangle = \sup_B \langle ft, t_\kappa\rangle = 0$, then $\langle f^B, u\rangle = 0$ and $\langle f, \alpha\rangle = 0$. We will complete the proof by showing that $\gamma \in G$ with $\langle f, \gamma\rangle = 0$ implies $\gamma \in h_G(BCV)$ i.e., $BCV \in \gamma$. Otherwise $T - BCV \in \gamma$, and $\gamma \in h(T - BCV)$. If (s_ν) is a net in $T - BCV$ such that $s_\nu \to \gamma$ in βT, then $\langle f, s_\nu\rangle \to \langle f, \gamma\rangle$. But $\langle f, s_\nu\rangle = 1$ for all ν implies $\langle f, \gamma\rangle = 1$, a contradiction. This completes the proof.

The next set of comments requires some distinction between the abelian and non-abelian cases. First consider the left transformation group (G, M) with composition as the action. If T is abelian, (G, M) is minimal. For if $m \in M$, and $t \in T$, then $ut = utu \in G$ and $mt = umt = utm$. Since $\{mt \mid t \in T\}$ is dense in M, this shows our assertion. When T is non-abelian, the situation is somewhat different. In this case, we know from the general theory that there exists a G-almost periodic point $p \in M$. Moreover, since (G,M,T) is a bi-transformation group, then $\{pt \mid t \in T\}$ is a dense set of G-almost periodic points in M, and thus the G-almost periodic points $A(G,M)$ satisfy $\mathrm{cls}(A(G,M)) = M$. Finally, there are G-almost periodic idempotents. For if $pv = p$ and $qp = v$, then $(qu)p = qp = v$ and $qu \in G$, whence v is G-almost periodic. It is an interesting open question whether u is also G-almost periodic. To take into account the fact that, in general, we can no longer fix the idempotent to be u, we need the following observations about the groups $Gv(v \in J$, the idempotents of $M)$. Recall that in the general algebraic theory, the idempotent was arbitrarily chosen. Since we could carry out the general theory for Gv, where $v \in J$, it makes sense to speak of the $\tau(\mathscr{R})$ $(= \tau(\mathscr{R}v) = \tau(\mathscr{R}_v))$ topology on Gv.

LEMMA (3.5). *Let $w, v \in J$. Then $\phi:(Gw,\tau(\mathscr{R})) \to (Gv,\tau(\mathscr{R}))$, $\alpha w \to \alpha v$, is an isomorphism.*

PROOF. Clearly ϕ is bijective. Thus, we need only show that ϕ is continuous. Let $\alpha w \in \mathrm{cls}_{\tau(\mathscr{R})}(Kw)$. Then $f\alpha w \leqq (f^{Kw})w\,(f \in \mathscr{R}_R = \mathscr{R} \cap \mathscr{C}_R)$. Pick $g \in \mathscr{R}_R$ and $p \in \beta T$. Then $\langle (g\alpha)v, p\rangle = \langle g\alpha wv, p\rangle = \langle g\alpha w, vp\rangle \leqq \langle (g^{Kw})w, vp\rangle = \langle (g^{Kvw})w, vp\rangle$ $\leqq \langle (g^{Kv})w^2, vp\rangle$ [2, 4 of 11.7] $= \langle (g^{Kv})w, vp\rangle = \langle (g^{Kv})wv, p\rangle = \langle (g^{Kv})v, p\rangle$. Thus, $g\alpha v \leqq (g^{Kv})v$, from which it follows that $\alpha v = \phi(\alpha w) \in \mathrm{cls}_{\tau(\mathscr{R})}(Kv) = \mathrm{cls}_{\tau(\mathscr{R})}(\phi(Kw))$.

The content of (3.5) is that analysis is identical on each group $Gv(v \in J)$. In passing, one should note that the $\mathscr{R}_v$-topology on M[3, Def. (2.2)] coincides with the $\mathscr{R}_w$-topology on $M\,(v, w \in J)$.

LEMMA (3.6). *Let $p \in A(G,M)$ and $v \in J$ with $pv = p$. Let $B \in p$. Then $(\mathrm{int}_{\tau(\mathscr{R})}\mathrm{cls}_{\tau(\mathscr{R})}h_{Gv}(B)) \cap h_{Gv}(B) \neq \emptyset$, where $h_{Gv}(B) = h(B) \cap Gv$.*

PROOF. Since $B \in p = pv$, then $p \in h(B) \cap Mv = h(B) \cap Gv$. Thus $Gv \subset \mathrm{cls}_M(Gp)$ and $h(B)$ an ordinary neighborhood of p implies that $F \cdot h(B) \supset Gv$ for some finite subset F of G. Then $F \cdot (h(B) \cap Gv) = F \cdot h_{Gv}(B) \supset Gv$, from which it follows that $\mathrm{int}_{\tau(\mathscr{R})}\mathrm{cls}_{\tau(\mathscr{R})}h_{Gv}(B) \neq \emptyset$. This of course implies that $(\mathrm{int}_{\tau(\mathscr{R})}\mathrm{cls}_{\tau(\mathscr{R})}h_{Gv}(B)) \cap h_{Gv}(B) \neq \emptyset$.

In preparation for the key lemma to obtain our results, we need the following observations. We have noted that the canonical homomorphism from T into $B(T,\mathscr{T}) \simeq (G/H(G,\mathscr{R}),\tau(\mathscr{R}))$ is $t \to H(G,\mathscr{R})(utu)$. This leads to the natural extension $\phi:\beta T \to B(T,\mathscr{T}), \phi(p) = H(G,\mathscr{R})(upu)$. Now let χ be a finite-dimensional continuous representation of $(T,\mathscr{T})$. Then of course, χ has an extension

$\bar{\chi}$ to $B(T,\mathcal{T})$ satisfying the equations $\bar{\chi}(\phi(t)) = \bar{\chi}(H(G,\mathcal{R})utu) = \chi(t) (t \in T)$. In addition, we have already defined the extension $\hat{\chi}$ of χ to βT (see (3.2)). Since $\bar{\chi}\phi \mid T = \chi = \hat{\chi} \mid T$, we have by continuity that $\bar{\chi}\phi = \hat{\chi}$, i.e., $\bar{\chi}(\phi(p)) = \hat{\chi}(p) \, (p \in \beta T)$.

LEMMA (3.7). *Suppose that* $u \in A(G,M)$. *Let* $B \subset T$, *and assume that* $h_G(B) \neq \emptyset$. *Then there exist continuous finite-dimensional unitary representations* $\chi_1, \cdots, \chi_n$ *of* $(T,\mathcal{T})$, $\varepsilon > 0$, *and* $\alpha \in h_G(B)$ *such that if* $\| \chi_i(t) - I \| < \varepsilon$ $(1 \leqq i \leqq n)$, *then* $BCV \in \alpha tu$ *for every neighborhood* $V \in \mathcal{N}_e(\mathcal{T})$ *and every* $C \in u$.

PROOF. Since $h_G(B) \neq \emptyset$, it follows that there exists $\alpha \in h_G(B)$ such that $\alpha \in \mathrm{int}_{\tau(\mathcal{R})} \mathrm{cls}_{\tau(\mathcal{R})} h_G(B)$.

Now recall that $B(T,\mathcal{T}) = (G/H(G,\mathcal{R}), \tau(\mathcal{R}))$. We claim that if N is a $\tau(\mathcal{R})$-open neighborhood of γ, then there exists continuous finite-dimensional unitary representations $\chi_1, \cdots, \chi_n$ of $(T,\mathcal{T})$ and $\varepsilon > 0$ such that if $\| \hat{\chi}_i(\beta) - \hat{\chi}_i(\gamma) \| < \varepsilon$, then $\pi_1(\beta) \in \pi_1(N)$, where $\hat{\chi}_i$ is as in (3.2) and $\pi_1 : (G, \tau(\mathcal{R})) \xrightarrow{\sim} (G/H(G,\mathcal{R}), \tau(\mathcal{R}))$ is the canonical map. First note that $\phi \mid G = \pi_1$, where ϕ is the above mentioned map. Now $\pi_1(N) = \phi(N)$ is an open neighborhood of $\pi_1 \gamma$ in $B(T,\mathcal{T})$. Then choose finite-dimensional unitary representations $\chi_1, \cdots, \chi_n$, and $\varepsilon > 0$ such that $\| \bar{\chi}_i(\pi_1\beta) - \bar{\chi}_i(\pi_1\gamma) \| < \varepsilon$ $(i = 1, \cdots, n)$ implies $\pi_1\beta \in \pi_1 N$. Since $\| \hat{\chi}_i(\beta) - \hat{\chi}_i(\gamma) \| = \| \bar{\chi}_i(\pi_1\beta) - \bar{\chi}_i(\pi_1\gamma) \|$, this means that $\| \hat{\chi}_i(\beta) - \hat{\chi}_i(\gamma) \| < \varepsilon$ $(i = 1, \cdots, n)$ implies $\pi_1\beta \in \pi_1 N$, as asserted.

Next, we apply the above paragraph to $L = \mathrm{int}_{\tau(\mathcal{R})} \mathrm{cls}_{\tau(\mathcal{R})} h_G(B)$, to produce $\chi_1, \cdots, \chi_n$, $\varepsilon > 0$ such that $\| \hat{\chi}_i(\beta) - \hat{\chi}_i(\alpha) \| < \varepsilon$ implies $\pi_1(\beta) \in \pi_1(L)$. Now let $t \in T$ with $\| \chi_i(t) - I \| < \varepsilon$ $(1 \leqq i \leqq n)$. Then $\| \hat{\chi}_i(\alpha tu) - \hat{\chi}_i(\alpha) \| = \| \hat{\chi}_i(\alpha) \hat{\chi}_i(tu) - \hat{\chi}_i(\alpha) \| \leqq \| \hat{\chi}_i(\alpha) \| \, \| \hat{\chi}_i(tu) - \hat{\chi}_i(u) \| \leqq \| \hat{\chi}_i(t) - I \| \, \| \hat{\chi}_i(u) \| \leqq \| \hat{\chi}_i(t) - I \| < \varepsilon$ $(1 \leqq i \leqq n)$, since χ_i is unitary. Thus, $\pi_1(\alpha tu) \in \pi_1(L)$, and $\alpha tu \in H(G,\mathcal{R}) \mathrm{cls}_{\tau(\mathcal{R})} L = \mathrm{cls}_{\tau(\mathcal{R})} L$, by [2, Lemma 14.7]. Now $\mathrm{cls}_{\tau(\mathcal{R})} L \subset \mathrm{cls}_{\tau(\mathcal{R})} h_G(B) \subset h_G(BCV)$ for every $V \in \mathcal{N}_e(\mathcal{T})$ and $C \in u$ (3.4). Thus, $BCV \in \alpha tu$, as desired.

Note that if $v \in J$, $v \in A(G,M)$ and $h_{Gv}(B) \neq \emptyset$, then (3.7) holds with $BCV \in \alpha tv$ for every $V \in \mathcal{N}_e(\mathcal{T})$ and $C \in v$.

The next result constitutes the major theorem of this section. We will derive several consequences which will require distinguishing between the commutative and non-commutative cases, and which in turn will lead to several generalizations of [4].

DEFINITION (3.8). Let $A \subset T$. Then A is *big* if there exists a minimal right ideal N of βT such that $h(A) \cap N \neq \emptyset$.

One should note that as with the idempotent u, the minimal ideal M was chosen arbitrarily. Thus, we could have performed all of our analysis in any minimal right ideal. Thus, if A is big, there is no loss of generality in assuming $h(A) \cap M \neq \emptyset$. Since $h(A) \cap M$ is open in M, $h(A) \cap A(G,M) \neq \emptyset$. If $p \in A(G,M) \cap h(A)$ and $pv = p$, then $h_{Gv}(A) \neq \emptyset$, and $v \in A(G,M)$.

THEOREM (3.9). *Let $B \subset T$ such that B is big. Then there exist continuous finite-dimensional unitary representations $\chi_1, \cdots, \chi_n$ of $(T, \mathscr{T})$, $\varepsilon > 0$, and v a minimal idempotent such that if $\| \chi_i(t) - I \| < \varepsilon \, (1 \leqq i \leqq n)$, then $t \in B^{-1}BCVC^{-1}$ for every $V \in \mathscr{N}_e(\mathscr{T})$ and $C \in v$.*

PROOF. By the above comment, there exists $v^2 = v \in A(G,M)$ with $h_{Gv}(B) \neq \emptyset$. By (3.7), there exists $\alpha \in h_{Gv}(B)$ such that $BCV \in \alpha t v$ for every $V \in \mathscr{N}_e(\mathscr{T})$ and $C \in v$. Thus, $(BCV)(\alpha t) \in v$. Now it is direct to verify that $(BCV)(\alpha t) = t^{-1}[(BCV)\alpha]$ Thus $t^{-1}[(BCV)\alpha] \in v$. Moreover, if $s \in (BCV)\alpha$, then $(BCV)s^{-1} \in \alpha$. Since $B \in \alpha$, $(BCV)s^{-1} \cap B \neq \emptyset$, whence $s \in B^{-1}BCV$. Thus $(BCV)\alpha \subset B^{-1}BCV$. Now $C \in v$ implies $t^{-1}[(BCV)\alpha] \cap C \neq \emptyset$. Hence $t^{-1}[B^{-1}BCV] \cap C \neq \emptyset$, from which it follows that $t \in B^{-1}BCVC^{-1}$, as desired.

The first application of (3.9) is to discretely syndetic sets.

LEMMA (3.10). *Let C be discretely syndetic, and v be a minimal idempotent. Then there exists $c \in C$ with $Cc^{-1} \in v$.*

PROOF. We claim that $Cv \cap C \neq \emptyset$. For if $Cv \cap C = \emptyset$, then $\emptyset = (Cv \cap C)v = Cv^2 \cap Cv = Cv \cap Cv = Cv$, which is a contradiction [2, Lemma (8.15)]. Thus $c \in Cv$ for some $c \in C$. This means that $Cc^{-1} \in v$.

Note that in general, if $p \in N$, a minimal ideal, then $Cs \in p$ for some $s \in T$.

The next result is the non-abelian version of Følner's result.

COROLLARY (3.11). *Let C be discretely syndetic. Then there exist continuous finite-dimensional unitary representations $\chi_1, \cdots, \chi_n$ of $(T, \mathscr{T})$ and $\varepsilon > 0$ such that if $\| \chi_i(t) - I \| < \varepsilon \; (1 \leqq i \leqq n)$, then $t \in C^{-1}CCVC^{-1}$ for every $V \in \mathscr{N}_e(\mathscr{T})$.*

PROOF. Since $\operatorname{cls} A(G,M) = M$, and $h(C)$ is open, there exists $p \in A(G,M)$ with $p \in h(C)$. If $v \in J$ with $pv = p$, then $v \in A(G,M)$, as previously noted, and $p \in h_{Gv}(C)$. Choose $c \in C$ with $Cc^{-1} \in v$ by (3.10). Applying (3.9) with $B = C$, $C = Cc^{-1}$, the set involved becomes $C^{-1}C \; Cc^{-1}V(Vc^{-1})^{-1} = C^{-1}CCc^{-1}VcC^{-1}$. Since V is an arbitrary neighborhood of e, so is $c^{-1}Vc$. The result follows.

One should note that, in (3.11), the representations $\chi_1, \cdots, \chi_n$ and ε are a function of the syndetic set C alone, and do not depend on V. Note also that if C is discretely syndetic, then C is big; moreover, $h(C) \cap N \neq \emptyset$ for every minimal ideal N.

We now specialize (3.9) to the case when T is abelian. Recall that with this assumption, (G,M) is minimal.

THEOREM (3.12). *Let T be abelian and $A,B \subset T$ with $h_{Gv}(A) \cap h_{Gv}(B) \neq \emptyset$. Then there exist continuous characters $\chi_1,\cdots,\chi_n$ of $(T,\mathscr{T})$ and $\varepsilon > 0$ such that $|\chi_i(t) - 1| < \varepsilon$ $(1 \leqq i \leqq n)$ implies $t \in A^{-1}BCV$ for every $V \subset \mathscr{N}_e(\mathscr{T})$ and every $C \in v$.*

PROOF. It is easy to see that $h_{Gv}(A) \cap h_{Gv}(B) = h_{Gv}(A \cap B)$. By (3.6), there exists $\alpha \in (\mathrm{int}_{\tau(\mathscr{R})}\mathrm{cls}_{\tau(\mathscr{R})}h_{Gv}(A \cap B)) \cap h_{Gv}(A \cap B)$. Thus, $\alpha \in (\mathrm{int}_{\tau(\mathscr{R})}\mathrm{cls}_{\tau(\mathscr{R})}h_{Gv}(B)) \cap h_{Gv}(A \cap B)$. Since $v \in A(G,M)$, it follows by (3.7) that there exist continuous characters $\chi_1,\cdots,\chi_n$ of $(T,\mathscr{T})$ and $\varepsilon > 0$ such that $|\chi_i(t) - 1| < \varepsilon (1 \leqq i \leqq n)$ implies $BCV \in \alpha tv$ for every $V \in \mathscr{N}_e(\mathscr{T})$ and $C \in v$. Moreover, $atv = \alpha vt = \alpha t$ implies $(BCV)^{-1} \in \alpha$. Since $A \in \alpha$, $(BCV)t^{-1} \cap A \neq \emptyset$, and thus $t \in A^{-1}BCV$ as desired.

COROLLARY (3.13).

1. *Let T be abelian, $v \in J$ and A, $B \subset T$ with $A \in v$, $B \in v$. Then there exists continuous characters $\chi_1,\cdots,\chi_n$ of $(T,\mathscr{T})$ and $\varepsilon > 0$ such that $|\chi_i(t) - 1| < \varepsilon$ and $(1 \leqq i \leqq n)$ implies $t \in A^{-1}BCV$ for every $V \in \mathscr{N}_e(\mathscr{T})$ and $C \in v$.*
2. *Let T be abelian and D discretely syndetic. Then there exist continuous, characters $\chi_1,\cdots,\chi_n$ of $(T,\mathscr{T})$ and $\varepsilon > 0$ such that if $|\chi_i(t) - 1| < \varepsilon(1 \leqq i \leqq n)$ then $t \in cls(D^{-1}DDs)$ for some $s \in D^{-1}$. Most generally, if $v \in J$, then $t \in \cap \{cls\,(D^{-1}DDs^{-1}) | s \in Dv\}$ (note by (3.10) that $Dv \cap D \neq \emptyset$).*

PROOF.

1. Immediate by (3.12).
2. Let $v \in J$, and consider $Dv \neq \emptyset$. Fix some element $r \in Dv$. Then $Dr^{-1} \in v$. Now apply (1) with $A = B = Dr^{-1}$ to get characters $\chi_1,\cdots,\chi_n$ and $\varepsilon > 0$. Now suppose that $s \in Dv$. Then $Ds^{-1} \in v$, and applying (1) with $C = Ds^{-1}$ will yield the desired result.

Note in (2) that the characters $\chi_1,\cdots,\chi_n$ depend on the *fixed* set Dr^{-1} and do not vary with $s \in Dv$. Also note that in addition to $Dv \neq \emptyset$, we actually have that Dv is big. For, $h(D) \cap M$ open means that $p \in h(D)$ and $D \in p$ for some $p \in M$. If $v \in J$, then $vp = p$, and $D \in vp$ implies $Dv \in p$.

REMARK (3.14).

1. If T is discrete abelian, then the set involved in (3.12) is $A^{-1}BC$. Thus, if D is discretely syndetic, then $t \in D^{-1}DDs$ for some $s \in D^{-1}$.

2. Suppose we have 2 discretely syndetic subsets D, K. Then Dr, $Ks \in u$ with $r \in D^{-1}$, $s \in K^{-1}$ by (3.10). Letting $A = Dr = C$, $B = Ks$, the set in 2 of (3.13) can be replaced by $\mathrm{cls}(D^{-1}KDs)$. An obvious extension holds for 3 discretely syndetic sets D, K, L.

3. Let D be any subset of T. Then either $D \in u$ or $D' \in u$, i.e., either D or D' is big. Thus, the conclusion of 1 of (3.13) holds for either $D^{-1}DDV$ or $(D')^{-1}(D')(D')V$, where $V \in \mathcal{N}_e(\mathcal{T})$.

4. We also claim that in the case that T is abelian, we can use A^{-1} if $A \in u$. This follows from the fact that if $\bar{u} = \{B^{-1} \mid B \in u\}$, then u is also a minimal idempotent and $A^{-1} \in \bar{u}$. We omit the details.

Applications of (3.13) yield some interesting characterizations of maximally almost periodic and minimally almost periodic topological groups when T is abelian.

THEOREM (3.15)

1. *$(T,\mathcal{T})$ is maximally almost periodic iff whenever $a \neq e$, then there exist B, C discretely syndetic subsets of T such that $a \notin \mathrm{cls}(C^{-1}CBb^{-1})$ for some $b \in \bigcup \{Bv \mid v \in J\}$.*

2. *$(T,\mathcal{T})$ is minimally almost periodic iff $\mathrm{cls}(A^{-1}BC) = T$ for all A, B, C discretely syndetic.*

PROOF,

1. Suppose $a \notin \mathrm{cls}(C^{-1}CBb)$ for some $b \in Bv$ and $v \in J$. Then by modifications noted in 2 of (3.14), there exists $\varepsilon > 0$ and a continuous character χ of $(T,\mathcal{T})$ such that $|\chi(a) - 1| \geqq \varepsilon$. The converse follows from [4, Corollary 1].

2. Suppose $(T,\mathcal{T})$ is minimally almost periodic and A, B, C discretely syndetic with $\mathrm{cls}(A^{-1}BC) \neq T$. Now again by 2 of (3.14), there exists $s \in T$, $\varepsilon > 0$, and continuous characters $\chi_1, \cdots, \chi_n$ such that $|\chi_i(t) - 1| < \varepsilon$ implies $t \in \mathrm{cls}(A^{-1}BCs)$. Then $\mathrm{cls}(A^{-1}BCs) = [\mathrm{cls}(A^{-1}BC)]s \neq T$ and thus there exists $b \in T$ and a continuous character χ such that $|\chi(b) - 1| \geqq \varepsilon$. This is a contradiction. So $\mathrm{cls}(A^{-1}BC) = T$. The converse follows from [4, Corollary 2].

The next few results show that the discretely syndetic subsets of T actually determine the topology of the Bohr compactification. Recall that the set of all

$(U,V) = \{\gamma \mid \gamma \in G$ and $U\gamma \cap V \neq \emptyset\}$, with $U, V \in u$ and $Vu = V$, yields a neighborhood base for u in (G,τ) [2, Prop. 11.14.1].

LEMMA (3.16). *Let T be abelian, $C \subset T$ with $C \in u$ and $Cu = C$. Then $t \in CCC^{-1}C^{-1}$ iff $ut \in (C,C)(C,C)^{-1}$.*

PROOF. Suppose $ut \in (C,C)(C,C)^{-1}$. Then $ut\alpha \in (C,C)$ for some $\alpha \in (C,C)$. Since T is abelian, $ut\alpha = \alpha t$, and $\alpha t \in (C,C)$. This means that $C(\alpha t) \cap C \neq \emptyset$. Since $Cc^{-1} \in \alpha t$ for some $c \in C$, then $CC^{-1} \in \alpha t$, or $CC^{-1}t^{-1} \in \alpha$. Since $\alpha \in (C,C)$, then $CC^{-1} \in \alpha$, from which $CC^{-1}t^{-1} \cap CC^{-1} \neq \emptyset$ and $t \in CCC^{-1}C^{-1}$. Conversely, $t \in CCC^{-1}C^{-1}$ implies $Cs^{-1} \cap C \neq \emptyset$, $C(ts)^{-1} \cap C \neq \emptyset$ for some $s \in T$. Since $(Cu)s^{-1} = C(us)$, then $C = Cu$ implies $(Cu)s^{-1} \cap C = C(us) \cap C \neq \emptyset$, and $C(uts) \cap C \neq \emptyset$, giving the desired result.

THEOREM (3.17). *Let T be discretely abelian. Regarding $T \subset G$ (i.e., identifying t with utu), $\{\mathrm{cls}_\tau \pi(CCC^{-1}C^{-1}) \mid C$ discretely syndetic$\}$ is a neighborhood base for $\pi(u)$ in $(G/E,\tau)$.*

PROOF. Recall that $\pi : (G,\tau) \to (G/E,\tau)$. By (3.11), each set $\mathrm{cls}_\tau \pi(CCC^{-1}C^{-1})$ with C discretely syndetic is a neighborhood of $\pi(u)$.

We next show that if $N \in \mathcal{N}_{\pi(u)}(\tau)$, there exists $A \in u$ with $Au = A$ and $\pi(A,A) \subset N$. Now there exists $U, V \in u$ with $Vu = V$ and $\pi(U,V) \subset N$. Now $W = U \cap V \in u$. By [3, Lemma 2.6], there exists $A \in u$ with $Au = A$ and $h(A) \subset h(W)$. Now $\alpha \in (A,A)$ implies $\alpha t \in h(A) \subset h(W)$ for some $t \in A = Au$. Thus, $ut \in h(A) \subset h(W)$ and $t \in Wu$, whence $\alpha \in (W, Wu)$. Since $(W, Wu) \subset (U, Vu) = (U,V)$, this yields the result. Note also that A is syndetic: if $p \in \beta T$, $Ap = (Au)p = A(up)$, and $up \in M$. Also, $q \in M$ implies $Aq \neq \emptyset$, since $Aq = \emptyset$ and $r \in M$ with $qr = u$ would imply $(Aq)r = A(qr) = Au = A = \emptyset$, a contradiction. Now use [2, 8.15].

Finally, choose K closed in $\mathcal{N}_{\pi(u)}(\tau)$. There exists $L \in \mathcal{N}_{\pi(u)}$ with $LL^{-1} \subset K$. By the above paragraph, there exists $C \in u$ with $Cu = C$ and $\pi(C,C) \subset L$. Then C is syndetic and $\pi[(C,C)(C,C)^{-1}] \subset LL^{-1} \subset K$. Then by (3.16), $\mathrm{cls}_\tau \pi(CCC^{-1}C^{-1}) = \mathrm{cls}_\tau \pi[(C,C)(C,C)^{-1}] \subset \mathrm{cls}_\tau K = K$. This completes the proof.

Using (3.14), one can produce some obvious modifications, e.g., replacing $CCC^{-1}CC^{-1}$ by $CC^{-1}s$ for some $s \in C^{-1}$ or $CBC^{-1}B^{-1}$ for B, C discretely syndetic.

We now extend this result to $(T, \mathscr{T})$.

COROLLARY (3.18). *Let T be abelian. Regarding* $T \subset G$, $\{\mathrm{cls}_{\tau(\mathscr{R})}\pi_1(CCC^{-1}C) \mid C$ discretely syndetic$\}$ is a neighborhood base for $\pi_1(u)$ in $(G/H(G,\mathscr{R}),\tau(\mathscr{R}))$, where $\pi_1 : (G,\tau(\mathscr{R})) \to (G/H(G,\mathscr{R}),\tau(\mathscr{R}))$.

PROOF. We have the following commutative diagram,

$$\begin{array}{cccc} & (G,\tau) & \overset{\pi}{\to} & (G/E,\tau) \\ T \overset{i\nearrow}{\underset{i\searrow}{}} & & & \searrow\phi \\ & (G,\tau(\mathscr{R})) & \overset{\pi_1}{\to} & (G/H(G,\mathscr{R}),\tau(\mathscr{R}) \end{array}$$

where t is identified with $ut = utu$. Now by (3.17), $\{\mathrm{cls}_\tau\pi(CCC^{-1}C^{-1}) \mid C$ discretely syndetic$\}$ is a neighborhood base of $\pi(u)$. Then since ϕ is a transformation group homomorphism between two almost periodic minimal transformation groups, ϕ is open [6, Theorem (8.1)], and thus both open and closed. Since $\phi\, \mathrm{cls}_\tau\pi(CCC^{-1}C^{-1}) = \mathrm{cls}_{\tau(\mathscr{R})}\phi\,\pi\,(CCC^{-1}C^{-1}) = \mathrm{cls}_{\tau(\mathscr{R})}\pi_1(CCC^{-1}C^{-1})$, the desired result is obtained.

One should note in passing that it was *not* necessary to use sets of the form $\mathrm{cls}_{\tau(\mathscr{R})}\pi_1(\mathrm{cls}_{\mathscr{T}}CC^{-1}C^{-1}U)$, where $U \in \mathscr{N}_e(\mathscr{T})$, as might be expected from (3.13). This is due to the fact that (3.12) and (3.13) are *stronger* statements than simply assertions about the Bohr compactification's topology. Indeed, it does not seem possible to recover (3.13) by assuming the result of (3.18). In fact, if this were the case, then one could use the result of (3.18) to prove an assertion like 1 of (3.13) without using neighborhoods of e in $(T,\mathscr{T})$. This would yield many obvious contradictions.

The original statement of Følner's result uses $\mathrm{Re}\,\chi_i(t) \geqq 0$, i.e., $\varepsilon = \pi/2$. The following result shows that this statement is equivalent to letting ε vary with $U \in \mathscr{N}_e(\mathscr{T})$ in the case that T is abelian.

LEMMA (3.19). *Let T be abelian and $\psi : T \to B(T,\mathscr{T})$ canonical. Then for erery $N \in \mathscr{N}_e(B(T,\mathscr{T}))$ there exists continuous characters $\chi_1,\cdots,\chi_n$ of T such that* $\mathrm{Re}\,\chi_i(t) \geqq 0$ $(i = 1,\cdots,n)$ *implies* $\psi(t) \in N$.

PROOF. Let $\mathscr{F}$ be the finite subsets of continuous characters of T, partially ordered by subset inclusion. If $F \in \mathscr{F}$, let $E_F = \{a \mid a \in B(T,\mathscr{T}),\ \mathrm{Re}\,\bar{\chi}(a) \geqq 0$ $(\chi \in F)\}$, where $\bar{\chi}$ is the extension of χ to B (see (3.7)). Since E_F is closed and $e \in E_F (F \in \mathscr{F})$, then $\{E_F \mid F \in \mathscr{F}\}$ is a closed filter base with the finite intersection property. We claim that $\cap E_F = \{e\}$, which will complete the proof. First note that $\{e\} = \{a \mid \bar{\chi}(a) = 1$ for every continuous character $\chi\}$. Now suppose $a \in B(T,\mathscr{F})$ and $\bar{\chi}(a) \neq 1$ for some continuous character χ. We consider two cases. First, if $\mathrm{Re}\,\bar{\chi}(a) < 0$, then $a \notin E_\chi$. Second, if $\mathrm{Re}\,\bar{\chi}(a) \geqq 0$ and $0 < \arg\bar{\chi}(a) \leqq \pi/2$, then choose N minimal such that $\arg(\bar{\chi})^N(a) > \pi/2$. Then $N > 1$ and $\arg\bar{\chi}(a) \leqq \arg(\bar{\chi})^{N-1}(a) \leqq \pi/2$, whence $\pi/2 < \arg(\bar{\chi})^N(a) = \arg(\bar{\chi})^{N-1}(a) + \arg\bar{\chi}(a) \leqq \pi$. If $3\pi/2 \leqq \arg\bar{\chi}(a) < 2\pi$, then $0 < \arg(\bar{\chi})^{-1}(a) \leqq \pi/2$ and by the above, $\pi/2 < \arg(\bar{\chi})^{-N}(a) \leqq \pi$ for some N. Since the other containment is obvious, the conclusion follows.

Note that $\psi = \pi\rho$, where $\rho: T \to G$, $t \to utu$. Replacing $|\chi_i(t) - 1| < \varepsilon$ by $\mathrm{Re}\,\chi_i(t) \geqq 0$ by using this comment and (3.18) in the first paragraphs of (3.7) and (3.9), we can state both of these results using the form $\mathrm{Re}\,\chi_i(t) \geqq 0$.

We now return to the non-abelian case. Some of the modifications still hold in this case. For example, if B, C are discretely syndetic, then Bb^{-1}, $Cc^{-1} \in v$ where $v \in A(G,M)$ and $b \in B$, $c \in C$. Applying (3.9) as in (3.11), we can replace $C^{-1}CCVC^{-1}$ by $B^{-1}BCVC^{-1}$. Also from (3.11), we can get $t \in \cap \{C^{-1}CCVC^{-1} \mid V \in \mathscr{N}_e\,\mathscr{T})\}$, which is *not* cls $(C^{-1}CCC^{-1})$ in general. Moreover, if D is any subset of T, then (3.9) will hold for either $D^{-1}DDVD^{-1}$ or $(D')^{-1}(D')(D')V(D')^{-1}$ as in (3.14). Using the fact that if χ is a continuous finite-dimensional unitary representation of $(T,\mathscr{T})$, then for every $\varepsilon > 0$, $\{t \mid \|\chi(t) - I\| < \varepsilon\}$ is discretely syndetic, we have the following result which is similar to (3.15).

THEOREM (3.20).

1. *$(T,\mathscr{T})$ is maximally periodic iff whenever $a \neq e$, then there exist B, C discretely syndetic subsets of T and $V \in \mathscr{N}_e(\mathscr{T})$ with $a \notin B^{-1}BCVC^{-1}$.*
2. *$(T,\mathscr{T})$ is minimally almost periodic iff $B^{-1}BCVC^{-1} = T$ for all B, C discretely syndetic and $V \in \mathscr{N}_e(\mathscr{T})$.*

One case in which $(T,\mathscr{T})$, with T not necessary abelian, satisfies the hypothesis of this section is when $\mathscr{R}_u \subset \mathscr{K} = \{f \mid ft \in \mathfrak{U}(u)\ (t \in T)\}$. To see this, recall by (2.8) that since $H(G,\mathscr{R}) \subset ER$, it is sufficient to show that $H(G,\mathscr{R}) \supset ER$. Now since $\tau(\mathscr{R}_u) \subset \tau(\mathscr{K})$, then $H(G,\mathscr{R}) \supset \{\mathrm{cls}_{\tau(\mathscr{K})} V \mid V \text{ a } \tau(\mathscr{K})\text{-neighborhood of } u\} = E$, by [2, Prop. 15.13]. Thus, $H(G,\mathscr{R}) \supset ER$, and the desired equality is obtained.

4. Another criterion for maximal almost periodicity

In [1], it was proved that if $(T,\mathscr{T})$ is a topological group with T abelian, and $s \neq e$, there exists a continuous character χ with $\chi(s) \neq 1$ iff for some discretely syndetic open symmetric neighborhood U of e, $s \notin U^6$. By using Følner's result, it is easy to replace U^6 by U^5. Our results enable us to obtain U^4, THEOREM (4.1). *Let $(T,\mathscr{T})$ be an abelian topological group and $s \neq e$. Then there exists a continuous character χ with $\chi(s) \neq 1$ iff there exists a symmetric open discretely syndetic neighborhood U of e with $s \notin U^4$.*

PROOF. Suppose such a U exists. Since U is discretely syndetic, there exists $w \in U^{-1} = U$ with $Uw \in u$. There exists $V \in \mathscr{N}_e(\mathscr{T})$ with $wV \subset U$. Now apply 1. of (3.13) to the set $(Uw)^{-1}(Uw)(Uw)V = U^{-1}U^2wV \subset U^4$. Since $s \notin U^4$, there exists a continuous character χ with $\chi(s) \neq 1$. The converse follows from the original result.

It was also shown in [1] that U^3 is the best result that could be obtained. The authors conjecture that U^3 is indeed obtainable by extensions of the methods used in this paper. Note that in (4.1), we can replace U^4 by $\mathrm{cls}(U^3w)$ for any $w \in Uu$. For, $s \notin \mathrm{cls}\, U^3w$ implies $s \notin U^3wV = (Uw)^{-1}Uw)(Uw)V$ for some $V \in \mathscr{N}_e(\mathscr{T})$, and, again, we can apply 1. of (3.13).

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